Stability of large periodic solutions of Klein-Gordon near a homoclinic orbit

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STABILITY OF LARGE PERIODIC SOLUTIONS OF KLEIN-GORDON NEAR A HOMOCLINIC ORBIT

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Abstract. — We consider the Klein-Gordon equation (KG) on a Riemannian surface M

$$\partial_t^2 u - \Delta u - m^2 u + u^{2p+1} = 0, \quad p \in \mathbb{N}^*, \quad (t,x) \in \mathbb{R} \times M,$$

which is globally well-posed in the energy space. Viewed as a first order Hamiltonian system in the variables $(u, v \equiv \partial_t u)$, the associated flow lets invariant the two dimensional space of (u,v) independent of x. It turns out that in this invariant space, there is a homoclinic orbit to the origin, and a family of periodic solutions inside the loops of the homoclinic orbit. In this paper we study the stability of these periodic orbits under the (KG) flow, i.e. when turning on the nonlinear interaction with the non stationary modes. By a shadowing method, we prove that around the periodic orbits, solutions stay close to them during a time of order $(\ln \eta)^2$, where η is the distance between the periodic orbit considered and the homoclinic orbit.

1. Introduction, statement of the main results

1.1. General introduction. — Denote by M a compact Riemannian manifold without boundary of dimension 1,2 or 3 and denote by $\Delta = \Delta_M$ the Laplace-Beltrami operator on M. In this paper we study the following nonlinear Klein-Gordon (KG) equation

$$\text{(KG)} \qquad \begin{cases} \partial_t^2 u - \Delta u - m^2 u + u^{2p+1} = 0, \quad (t,x) \in \mathbb{R} \times M, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases}$$

where $p \geq 1$ is an integer, and $(u_0, u_1) \in H^1(M) \times L^2(M)$ are real-valued.

When M is the torus, this is the usual Klein-Gordon equation with periodic boundary conditions. For a more general manifold M, equation (KG) can model a vibrating membrane, for example the earth during an earthquake.

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Let us recall that there exists a Hilbert basis of $L^2(M)$ composed with eigenfunctions $(e_n)_{n\geq 0}$ of Δ. Moreover, there exists a sequence $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \ldots$ so that

$$-\Delta e_n = \lambda_n^2 e_n, \quad n \geq 0.$$

We make following assumptions

Assumption 1. — *The parameter m satisfies $0 < m < \lambda_1$.*

Assumption 2. — *The manifold M and the integer p satisfy either:*

- *M is any compact manifold without boundary of dimension 1 or 2 and $p \geq 1$*
- *M is any compact manifold without boundary of dimension 3 and $p = 1$.*

Moreover, up to a rescaling, we can assume that $\operatorname{Vol} M = 1$.

Let us recall the well posedness result proved in [**3**]

Proposition 1.1 **([3], Theorem 2.2).** — *Under Assumptions* (1) *and* (2)*, the equation* (KG) *is globally well posed in the energy space $H^1(M) \times L^2(M)$.*

Under Assumptions 1 and 2, the stationary solutions of (KG) (solutions which only depend on the space variable) are exactly the constants $u = 0$, $u = m^{1/p}$ and $u = -m^{1/p}$. The origin is an equilibrium with an unstable direction. In fact, the eigenvalues of $-\Delta - m^2$ are the $(\lambda_k^2 - m^2)_{k\in\mathbb{N}}$. Since $0 < m < \lambda_1$, the case $k = 0$ only, gives the hyperbolic directions, corresponding to the solution $\exp(mt)$ for $t > 0$ (resp. $\exp(-mt)$ for $t < 0$). It turns out that (KG) admits a homoclinic orbit to the origin which is independent of x. In the previous work [**3**] we have proved that (KG) admits a family of heteroclinic connections to the center manifold which are close to this homoclinic orbit. The expected picture is that we have a tube of heteroclinic connections surrounding the homoclinic orbit (but the statement in [**3**] is not so precise, only a large family of heteroclinic orbits is constructed).

In many physical applications, the standard Klein-Gordon equation comes with a positive mass, namely

$$\partial_t^2 v - \Delta v + m^2 v + f(v) = 0, \quad (t, x) \in \mathbb{R} \times M, \tag{1.1}$$

in which case the linear part of the equation has no hyperbolic direction. In fact if u is a solution of (KG) then $v = u - m^{1/p}$ is a solution of (1.1) for some explicit nonlinearity f. So, when we consider (KG) instead of (1.1), we center the phase space at an unstable equilibrium ($u = 0$) instead of an elliptic one ($v = 0$). This is detailed in [**3**, Section 1.2].

As we pointed out in [**3**], the dynamics around the elliptic points $u = \pm m^{1/p}$ can be partially described by the KAM theory or the Birkhoff normal form theory. The KAM theory gives the existence of a large family of finite dimensional invariant tori close to the equilibrium (see [**7, 6, 5**] or the book [**5**]). The Birkhoff normal form approach gives the stability during polynomial times for any initial condition close to the equilibrium (see [**1**]).

In this work our point of view is different: (KG) also admits large periodic orbits inside the loops of the homoclinic orbit (see Figure 1) and we are interested in the stability of these large periodic orbits. Observe that they are not close to the origin since they turn around $u = m^{1/p}$. By a shadowing method, we prove that around the periodic orbits, solutions stay close to them during a time of order $(\ln\eta)^2$, where η is the distance between the periodic orbit considered and the homoclinic orbit. Actually, thanks to an energy method, it is easy to get a control for times of order $\ln\eta$, which is the typical timescale in the presence of a hyperbolic point. In our context, $\ln\eta$ is the timescale needed to achieve one loop, and our contribution consists in proving that we can follow the solution for $\ln\eta$ loops. The two main ingredients used in the proof are

- The Hamiltonian is negative [1] on the trajectory, which implies a confinement of the solution;
- The trajectory is close to the homoclinic orbit, which gives the pattern of the solution.

Combining these two facts, we conclude with a bootstrap argument.

1.2. Hamiltonian structure of (KG). — As in [**3**], we define the scalar product on $L^2(M)$ by $\langle f,g\rangle = \dfrac{1}{\operatorname{Vol} M}\displaystyle\int_M fg$, where $\operatorname{Vol} M$ denotes the volume of M, we assume that $\|e_n\|_{L^2}=1$ and we set $e_0=1$.

Denote by $v=\partial_t u$ and introduce

$$H = \frac{1}{2}\int_M \Big(|\nabla_x u|^2 + v^2 - m^2u^2\Big) + \frac{1}{2p+2}\int_M u^{2p+2}. \tag{1.2}$$

Then, the system (KG) is equivalent to

$$\dot u = \frac{\delta H}{\delta v}, \quad \dot v = -\frac{\delta H}{\delta u}. \tag{1.3}$$

We write

$$u(t,x) = \sum_{n=0}^{+\infty} a_n(t)e_n(x), \quad v(t,x) = \sum_{n=0}^{+\infty} b_n(t)e_n(x),$$

where

$$(a_n)_{n\in\mathbb{N}} \in h^1(\mathbb{N},\mathbb{R}) := \big\{x=(x_n)_{n\in\mathbb{N}} \mid \|x\|_{h^1}^2 = \sum_{n\in\mathbb{N}}(1+\lambda_n^2)|x_n|^2 < +\infty\big\},$$
$$(b_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{R}) := \big\{x=(x_n)_{n\in\mathbb{N}} \mid \|x\|_{\ell^2}^2 = \sum_{n\in\mathbb{N}}|x_n|^2 < +\infty\big\},$$

in such a way that to the continuous phase space $\mathcal{X} := H^1\times L^2$ corresponds the discrete one $h^1\times\ell^2$. We endow this space with the natural norm and distance

$$\begin{aligned}\|X\|_{\mathcal{X}} &= \|u\|_{h^1} + \|v\|_{\ell^2}, \quad \text{for} \quad X=(u,v)\\ \operatorname{dist}_{\mathcal{X}}(X,Y) &= \|X-Y\|_{\mathcal{X}}, \quad \text{for} \quad X,Y\in\mathcal{X}.\end{aligned} \tag{1.4}$$

In the coordinates $(a_n,b_n)_{n\geq 0}$, the Hamiltonian in (1.2) reads

$$H = \frac{1}{2}\sum_{n=0}^{+\infty}\Big[(\lambda_n^2-m^2)a_n^2 + b_n^2\Big] + \frac{1}{2p+2}\int_M\Big(\sum_{k=0}^{+\infty} a_k e_k(x)\Big)^{2p+2}\mathrm{d}x, \tag{1.5}$$

1. Our Hamiltonian H is defined so that $H(0)=0$.

and the system (1.3) becomes

$$\begin{cases} \dot{a}_n = b_n, \quad n \geq 0 \\ \dot{b}_n = -(\lambda_n^2 - m^2)a_n - \int_M \Big(\sum_{k=0}^{+\infty} a_k e_k(x)\Big)^{2p+1} e_n(x)\mathrm{d}x, \quad n \geq 0. \end{cases} \tag{1.6}$$

1.3. Space-stationary solutions: homoclinic orbit and family of periodic orbits. — The space-stationary solutions of (KG) exactly correspond to the solutions of (1.6) satisfying $a_n = b_n = 0$ for $n \geq 1$. In this case, the equation on (a_0, b_0) reads

$$\begin{cases} \dot{a}_0 = b_0 \\ \dot{b}_0 = m^2 a_0 - a_0^{2p+1}, \end{cases} \tag{1.7}$$

and this system possesses a homoclinic solution to 0, which we will denote in the sequel by $h : t \mapsto (\alpha(t), \beta(t))$. We denote by $\mathcal{K}_0 \subset h^1 \times \ell^2$ the curve which is described by $a_0(t) = h(t),\ b_0(t) = h'(t)$ and $a_n(t) = b_n(t) = 0$ for $n \geq 1$ (see Figure 1). Indeed we can explicitly compute

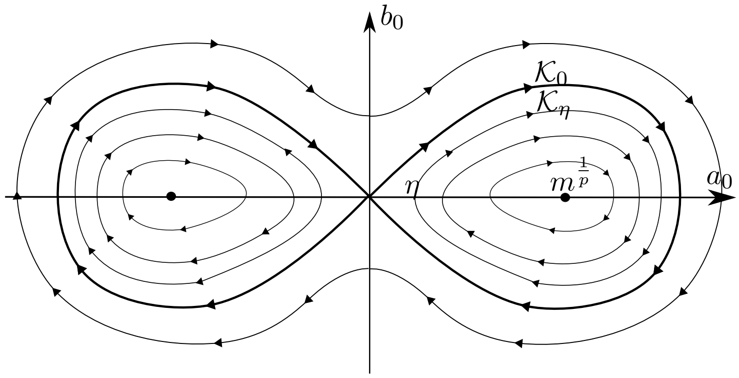

FIGURE 1. Phase portrait for the space-stationary set $a_n = b_n = 0$ for $n \geq 1$.

$$h(t) = \frac{m^{1/p}(p+1)^{1/(2p)}}{\big(\cosh(pmt)\big)^{1/p}}, \quad h'(t) = -m^{1/p+1}(p+1)^{1/(2p)} \frac{\sinh(pmt)}{\big(\cosh(pmt)\big)^{1/p+1}}. \tag{1.8}$$

For $\eta > 0$ denote by $\mathcal{K}_\eta$ the trajectory of (1.6) given by the initial conditions $a_0(0) = \eta$, $b_0(0) = 0$, $a_n(0) = b_n(0) = 0$ for $n \geq 1$ (see Figure 1). This orbit remains in the plane $\{(a, b) \in h^1 \times \ell^2 \mid a_n = b_n = 0 \text{ for } n \geq 1\}$ and is periodic with a period of order $\ln \dfrac{1}{\eta}$ (see below for the proof): most of that time is dedicated to cover the very principle and the very end of the loop, *i.e.* when (a_0, b_0) is close to $(\eta, 0)$.

Our main result states that for η small enough these periodic solutions are stable during a long time of order $(\ln \eta)^2$ in the sense that if one starts η^3-close to $\mathcal{K}_\eta$ then one remains η^2-close to $\mathcal{K}_\eta$ during a time of order $(\ln \eta)^2$. More precisely, recall the definition (1.4) of $\mathrm{dist}_{\mathcal{X}}$, then

Theorem 1.2. — *Let M, p and m fixed and satisfying Assumptions 1 and 2. There exists $\eta_0 > 0$, $0 < c < 1$ and $C > 0$ such that if $0 < \eta < \eta_0$ and if the initial datum $(a(0), b(0))$ satisfies*

$$\mathrm{dist}_{\mathcal{X}}\Big((a(0), b(0)), \mathcal{K}_\eta\Big) \leq \eta^3$$

then the solution of (1.6) *satisfies*

(i) There exists $T_\eta > 0$ satisfying $c \ln \frac{1}{\eta} \leq T_\eta \leq \frac{1}{c} \ln \frac{1}{\eta}$ such that

$$\mathrm{dist}_{\mathcal{X}}\Big((a(T_\eta), b(T_\eta)), (a(0), b(0))\Big) \leq C\eta^2$$

(ii) For all $|t| \leq c\big(\ln \frac{1}{\eta}\big)^2$

$$\mathrm{dist}_{\mathcal{X}}\Big(\big(a(t), b(t)\big), \mathcal{K}_\eta\Big) \leq C\eta^2.$$

This result shows that a trajectory issued from a tube of thickness η^3 centered on $\mathcal{K}_\eta$ describes in the mode 0 several loops around $(m^{\frac{1}{p}}, 0)$ and remains very small in the other modes. Indeed the time T_η is the time necessary for (a_0, b_0) to complete one loop around $(m^{\frac{1}{p}}, 0)$, and the previous result shows that the trajectory completes at least $c^2 \ln \frac{1}{\eta}$ loops around $(m^{\frac{1}{p}}, 0)$, staying inside the tube of thickness η^2 centered on $\mathcal{K}_\eta$.

Remark 1.3. — Numerical simulations confirm this stability result and seem to show that it holds true for larger timescales. Moreover, it is likely that there exist initial conditions as in Theorem 1.2 and which satisfy (*ii*) for all times. Such a result is out of reach with our method and could possibly be attacked with a KAM method.

1.4. On the linear case.— This stability problem is related to the more general question of existence of periodic or quasi-periodic solutions to (KG). This question is not evident even in the linear case. Actually, we can consider the linearized equation of (KG) around the periodic orbit $\mathcal{K}_\eta$ as a first approximation to understand the possible dynamics: take $a_0(t)$ as a periodic solution of the system (1.7) inside the loops (see Figure 1) and search for $w = \sum_{k\geq 1} a_k e_k$ as a (small) solution of the following linear wave equation

$$\partial_t^2 w - \Delta w - m^2 w + V(t) w = 0, \quad (t, x) \in \mathbb{R} \times M \tag{1.9}$$

where $t \mapsto V(t) = (2p+1) a_0^{2p}(t)$ is a time depending potential. In the discrete variables, the equation (1.9) reads

$$\begin{cases} \dot{a}_n = b_n \\ \dot{b}_n = -(\lambda_n^2 - m^2) a_n - V(t) a_n \end{cases} \quad \text{for } n \geq 1. \tag{1.10}$$

The existence of small periodic (or quasi-periodic) solutions of (1.9) is related to the reducibility of this equation to the autonomous case and to the Floquet theory. In the case where V is a small potential the question can be considered with the help of the KAM theory (see [**2**] in the case of a linear Schrödinger equation on a torus or [**4**] in the case of a linear Schrödinger equation on the line with a harmonic potential). To the best of our knowledge, there is no result in the case of a time dependent linear perturbation which is not small. With the same techniques, we can apply our result to the linear equation (1.7),(1.10), and this shows that there is at least some stability around the solutions $\mathcal{K}_\eta$.

Remark 1.4. — In our previous paper [**3**], we also consider the linearised system (1.10) but in the case where $t \mapsto a_0(t)$ describes the homoclinic orbit, and in which case the linear flow is bounded. We refer to [**3**, Appendix] where we crucially use that $\int_0^{+\infty} a_0(t)\mathrm{d}t < +\infty$.

1.5. Comments on the proof. — The method used in the proof of Theorem 1.2 relies on a bootstrap (or continuation) argument, is rather elementary and can be used in various situations. It can work when one tries to control perturbations of a one degree of freedom system, even when this perturbation has an infinite number of directions as in our case. The underlying idea is to give a precise description of one particular mode (here (a_0, b_0)) while the others are shown to remain small during the considered timescale. With these ingredients, we typically obtain a control of the solution during times of order $\ln \eta$. In order to improve the control up to order $(\ln \eta)^2$ we crucially use the fact that our periodic solution $\mathcal{K}_\eta$ is close to the homoclinic orbit. This enables us to define a modified functional $I(t)$ which controls the Sobolev norm of the exterior modes and which is almost constant (see Lemma 2.5).

Notations. — *In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or depend on the fixed quantities m and p.*
We denote by $\mathbb{N}$ the set of the non negative integers, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We set $\mathcal{X} = H^1(M) \times L^2(M)$.

2. Proof of the main result

2.1. Notations and strategy of the proof. — For $u = \sum_{k=0}^{+\infty} a_k e_k$ we define

$$U := \sum_{k=1}^{+\infty} a_k e_k, \tag{2.1}$$

so that in particular, u reads $u = a_0 e_0 + U$.

In the sequel we will use the following decomposition of the energy H (see (1.5))

$$H = \frac{1}{2}(b_0^2 - m^2 a_0^2) + J + \frac{1}{2(p+1)} \int_M \big(a_0 + U\big)^{2p+2} \mathrm{d}x = \frac{1}{2}b_0^2 + \frac{1}{2}f(a_0) + J + r, \tag{2.2}$$

where

$$f(x) := -m^2 x^2 + \frac{1}{p+1} x^{2p+2}, \tag{2.3}$$

$$J := \frac{1}{2} \sum_{k=1}^{+\infty} \Big[(\lambda_k^2 - m^2) a_k^2 + b_k^2\Big], \tag{2.4}$$

and

$$r(t) = \frac{1}{2(p+1)} \int_M \big((a_0 + U)^{2p+2} - a_0^{2p+2}\big) \mathrm{d}x. \tag{2.5}$$

Note that $J \sim \|U\|^2_{\mathcal{X}}$. In Subsection 2.4, we prove that J remains small. As a consequence r is also small since it is a quadratic quantity in U. Then, as long as J is small, the dynamics of the solution is governed by the space stationary dynamics (see Subsection 1.3) given by the Hamiltonian $H_0 = \frac{1}{2}b_0^2 + f(a_0)$.

Moreover, we introduce the notations q_k for $k \in \mathbb{N}$, defined by

$$q_k = \int_M \Big((a_0+U)^{2p+1} - a_0^{2p+1} - (2p+1)a_0^{2p}U\Big)e_k \mathrm{d}x, \tag{2.6}$$

so that the systems (1.6) read

$$\begin{cases} \dot{a}_0 = b_0 \\ \dot{b}_0 = -f'(a_0) - q_0, \end{cases} \tag{2.7}$$

$$\begin{cases} \dot{a}_k = b_k, \quad k \geq 1 \\ \dot{b}_k = -(\lambda_k^2 - m^2)a_k - (2p+1)a_0^{2p}a_k - q_k, \quad k \geq 1. \end{cases} \tag{2.8}$$

We finally fix a real $\delta \in]0, m^{\frac{1}{p}}[$, and denote by δ' the unique real of $]m^{\frac{1}{p}}, (p+1)^{\frac{1}{2p}}m^{\frac{1}{p}}[$ such that $f(\delta') = f(\delta)$.

2.2. Preliminary results. — In the sequel we will need

Lemma 2.1. — *Let $U \in H^1(M)$. Then*

- *When M has dimension 1 or 2, for all $2 \leq q < +\infty$*

$$\|U\|_{L^q(M)} \leq C_q J^{1/2}.$$

- *When M has dimension 3, for all $2 \leq q \leq 6$*

$$\|U\|_{L^q(M)} \leq C_q J^{1/2}.$$

Proof. — By Sobolev, in each of the previous cases, there exists $C_q > 0$ so that for all $U \in H^1(M)$ we have $\|U\|_{L^q(M)} \leq C_q \|U\|_{H^1(M)}$ and the result follows. $\square$

Lemma 2.2. — *Let $(a,b) \in \mathcal{X}$ such that* $\mathrm{dist}_{\mathcal{X}}\big((a,b), \mathcal{K}_\eta\big) \leq \eta^3$ *then*

$$H(a,b) = -\frac{1}{2}m^2\eta^2 + O(\eta^3). \tag{2.9}$$

Remark 2.3. — In the following, we prove in Lemma 2.6 a reciprocal result in the case J small.

Proof. — Let $(\tilde{a}, \tilde{b}) \in \mathcal{K}_\eta$ such that $\|(a,b) - (\tilde{a},\tilde{b})\|_{\mathcal{X}} \leq \eta^3$. On the one hand, since we have $(\eta, 0) \bigotimes_{n=1}^{+\infty}(0,0) \in \mathcal{K}_\eta$, then

$$H(\tilde{a},\tilde{b}) = H\Big((\eta,0)\bigotimes_{n=1}^{+\infty}(0,0)\Big) = -\frac{1}{2}m^2\eta^2.$$

On the other hand denoting by $\tilde{u} = \sum_{k=0}^{+\infty} \tilde{a}_k e_k$ and $\tilde{J} = \frac{1}{2}\sum_{k=1}^{+\infty}\Big[(\lambda_k^2 - m^2)\tilde{a}_k^2 + \tilde{b}_k^2\Big]$

$$\begin{aligned}
\Big|H(a,b) - H(\tilde{a},\tilde{b})\Big| &\leq \\
&\leq \frac{1}{2}|\tilde{b}^2 - b_0^2| + \frac{m^2}{2}|\tilde{a}_0^2 - a_0^2| + |J - \tilde{J}| + \frac{1}{2(p+1)}\int_M |u^{2p+2} - \tilde{u}^{2p+2}|\mathrm{d}x. \\
&\leq C\|(a,b) - (\tilde{a},\tilde{b})\|_{\mathcal{X}}\Big(\|(a,b)\|_{\mathcal{X}} + \|(\tilde{a},\tilde{b})\|_{\mathcal{X}}\Big) + C\|u - \tilde{u}\|_{L^{2p+2}(M)}\big(\|u\|^{2p+1}_{L^{2p+2}(M)} + \|\tilde{u}\|^{2p+1}_{L^{2p+2}(M)}\big) \\
&\leq C\|(a,b) - (\tilde{a},\tilde{b})\|_{\mathcal{X}} = O(\eta^3).
\end{aligned}$$

□

2.3. Estimates for J small. — We now fix an initial condition satisfying $\mathrm{dist}_{\mathcal{X}}\Big((a(0),b(0)),\mathcal{K}_\eta\Big) \leq \eta^3$ with $0 < \eta \ll 1$. In view of Lemma 2.2 we can choose η small enough in such a way that

$$H(a(0),b(0)) < 0. \tag{2.10}$$

By definition of $\mathcal{K}_\eta$ we have $a_0(0) > 0$ and without loss of generality we can also assume that $b_0(0) \geq 0$.

Lemma 2.4. — *For η small enough, as long as $J(t) \leq \eta^5$, there exists $C_1 = C_1(m) > 0$ such that*

$$\eta/2 \leq a_0(t) \leq C_1, \quad |b_0(t)| \leq C_1. \tag{2.11}$$

Proof. — Assume that $J \leq \eta^5$. First we prove that a_0 cannot vanish: Indeed if it was the case, by (2.2) we would have $H(a(t),b(t)) \geq 0$ for some $t \in \mathbb{R}$, which is in contradiction with (2.10) and the conservation of H. Next we prove that there exists $C > 0$ so that

$$\frac{1}{2(p+1)}\int_M \big(a_0 + U\big)^{2p+2}\mathrm{d}x \geq \frac{1}{4(p+1)}\,a_0^{2p+2} - CJ^{p+1}. \tag{2.12}$$

Write the binomial expansion

$$\int_M \big(a_0 + U\big)^{2p+2}\mathrm{d}x = a_0^{2p+2} + \sum_{j=1}^{2p+2}\binom{2p+2}{j}a_0^{2p+2-j}\int_M U^j\mathrm{d}x, \tag{2.13}$$

Apply Lemma 2.1 and use the Young inequality

$$c_1c_2 = (\varepsilon c_1)(\varepsilon^{-1}c_2) \leq \frac{\varepsilon^q}{q}c_1^q + \frac{1}{r\varepsilon^r}c_2^r, \quad c_1,c_2 \geq 0, \quad \varepsilon > 0, \quad \frac{1}{q} + \frac{1}{r} = 1,$$

to each term of the sum with $\varepsilon > 0$ small enough, and get

$$\sum_{j=1}^{2p+2}\binom{2p+2}{j}a_0^{2p+2-j}\Big|\int_M U^j\mathrm{d}x\Big| \leq C\sum_{j=1}^{2p+2}a_0^{2p+2-j}J^{j/2} \leq \frac{1}{2}a_0^{2p+2} + CJ^{p+1},$$

which together with (2.13) yields (2.12). Now, for $\eta > 0$ small enough, $CJ^{p+1} \leq J$ so by (2.12) and Lemma 2.2 we obtain

$$\frac{1}{2}(b_0^2 - m^2a_0^2) + \frac{1}{4(p+1)}\,a_0^{2p+2} \leq H = -\frac{1}{2}m^2\eta^2 + \mathcal{O}(\eta^3) \leq -\frac{1}{4}m^2\eta^2, \tag{2.14}$$

from which we deduce that $|b_0| \leq C_1$ and $0 < a_0 \leq C_1$ for some constant C_1 depending on m. Furthermore, the bound (2.14) implies $-m^2a_0^2/2 \leq H \leq -\frac{1}{4}m^2\eta^2$ which completes the proof. □

In the following we use the additional notations I and $\hat{r}$:

$$I := \frac{1}{2}\sum_{k=1}^{+\infty}\Big[\big(\lambda_k^2 - m^2 + (2p+1)a_0^{2p}\big)a_k^2 + b_k^2\Big],$$

$$\hat{r}(t) = \frac{1}{2(p+1)}\int_M \big((a_0+U)^{2p+2} - a_0^{2p+2} - (2p+1)a_0^{2p}U^2\big)\mathrm{d}x.$$

So that H reads

$$H = \frac{1}{2}b_0^2 + \frac{1}{2}f(a_0) + I + \hat{r}.$$

Observe that as long as $J \leq C\eta^5$, since a_0 is bounded (see (2.11)), there exists $K_0 > 0$ so that for all $(a_k, b_k)_{k\geq 1} \in h^1 \times \ell^2$,

$$J \leq I \leq K_0 J. \tag{2.15}$$

Lemma 2.5. — *As long as $J(t) \leq \eta^5$, we have the a priori bound*

$$|\dot{I}| \leq Ca_0^{2p-1}|\dot{a}_0|I + CI^{3/2}. \tag{2.16}$$

Proof. — We recall the definition (2.6) of the q_k, and that then the equations on (a_k, b_k) for $k \geq 1$ read

$$\dot{a}_k = b_k, \quad \dot{b}_k = -\big(\lambda_k^2 - m^2 + (2p+1)a_0^{2p}\big)a_k - q_k.$$

With this system, we compute

$$|\dot{I}| = \Big|p(2p+1)a_0^{2p-1}\dot{a}_0\sum_{k=1}^{+\infty}a_k^2 - \sum_{k=1}^{+\infty}b_kq_k\Big| \leq p(2p+1)a_0^{2p-1}|\dot{a}_0|\sum_{k=1}^{+\infty}a_k^2 + \|b\|_{\ell^2(\mathbb{N}^*)}\|q\|_{\ell^2(\mathbb{N}^*)}. \tag{2.17}$$

We have $\|b\|_{\ell^2(\mathbb{N}^*)} \leq CI^{1/2}$. Then by Parseval, (2.15) and the fact that $|a_0| \leq C$

$$\begin{aligned}
\|q\|_{\ell^2(\mathbb{N}^*)} &= \big\|(a_0+U)^{2p+1} - a_0^{2p+1} - (2p+1)a_0^{2p}U\big\|_{L^2(M)} \\
&\leq C\|U^2\|_{L^2(M)} + C\|U^{2p+1}\|_{L^2(M)} \\
&= C\|U\|_{L^4(M)}^2 + C\|U\|_{L^{2(2p+1)}(M)}^{2p+1} \\
&\leq CI + CI^{p+\frac{1}{2}} \leq CI, \qquad (2.18)
\end{aligned}$$

where in the last line we used Lemma 2.1. Then (2.18) together with (2.17) gives the result. □

Lemma 2.6. — *If $(a,b) \in \mathcal{X}$ such that $H(a,b) = -\frac{1}{2}m^2\eta^2 + O(\eta^3)$, $\eta \leq a_0$ and $J \leq \eta^3$, then*

$$\mathrm{dist}_{\mathcal{X}}\Big((a_0, b_0, 0, \cdots), \mathcal{K}_\eta\Big) \leq C\eta^2.$$

More precisely, $(a_0, b_0, 0, \cdots)$ is η^2-close to $(\tilde{a}_0, \tilde{b}_0, 0, \cdots) \in \mathcal{K}_\eta$ defined by

$$(\tilde{a}_0, \tilde{b}_0) = \begin{cases} \big(a_0, sgn(b_0)\sqrt{f(\eta) - f(a_0)}\big) & \text{if} \quad a_0 \in [\delta, \delta'], \\ \big(f^{-1}\big(f(\eta) - b_0^2\big), b_0\big) & \text{if} \quad a_0 \notin [\delta, \delta']. \end{cases}$$

Proof. — From the expression (2.2) of the energy, the assumption reads

$$b_0^2 + f(a_0) + J + r = -m^2\eta^2 + O(\eta^3).$$

On the one hand, observe that if $J \leq \eta^3$, then $r \leq c\eta^3$: the proof is similar to the one of (2.18). On the other hand, $f(\eta) = -m^2\eta^2 + O(\eta^3)$. Thus we have

$$b_0^2 + f(a_0) = f(\eta) + O(\eta^3). \tag{2.19}$$

Recall that $(\tilde{a}_0, \tilde{b}_0, 0, \cdots)$ belongs to $\mathcal{K}_\eta$ given that

$$\tilde{b}_0^2 + f(\tilde{a}_0) = f(\eta). \tag{2.20}$$

• Assume that $a_0 \notin [\delta, \delta']$. From (2.19) we deduce

$$a_0 = f^{-1}\Big(f(\eta) - b_0^2 + O(\eta^3)\Big).$$

Since $a_0 \geq \eta$, thus $|f'(a_0)| \geq c_0\eta$ and we can apply the mean value theorem to get

$$|\tilde{a}_0 - a_0| \leq C\eta^3 \sup |(f^{-1})'| \leq C\eta^2.$$

• In the region $a_0 \in [\delta, \delta']$, we define $g(b_0) = b_0^2$. Then we have $|g'(b_0)| \geq c$ and we can perform a similar argument to get $|\tilde{b}_0 - b_0| \leq C\eta^3$. □

2.4. First loop : proof of Theorem 1.2 (*i*). — This part is devoted to the proof of (*i*) of Theorem 1.2. More precisely, we prove a stronger version of it in Lemma 2.10, which will then be useful to prove (*ii*) of the theorem (in the next part).

Recall that in all this section we have fixed an initial condition such that

$$\mathrm{dist}_{\mathcal{X}}\Big((a(0), b(0)), \mathcal{K}_\eta\Big) \leq \eta^3 \tag{2.21}$$

and $0 < \eta \ll 1$ such that (2.10) is satisfied.

Lemma 2.7. — *There exists $K > 1$ independent of $\eta > 0$ such that: as long as $b_0(t) \geq 0$ and $|t| \leq \eta^{-2}$, J satisfies*

$$J(t) \leq KJ(0) \leq \eta^5.$$

Proof. — We prove it by a bootstrap argument. As long as $J(t) \leq \eta^5$ and $b_0(t) \geq 0$, the estimate (2.16) holds true and by Gronwall we get

$$\begin{aligned} J(t) \leq I(t) &\leq I(0)\mathrm{e}^{\int_0^{|t|}(\dot{a}_0(s)a_0^{2p-1}(s)+\eta^{5/2})\mathrm{d}s} \\ &\leq I(0)\mathrm{e}^{(a_0(t)^{2p}-a_0(0)^{2p}+|t|\eta^{5/2})} \\ &\leq I(0)\exp\big(2C_1^{2p}(m)+1\big), \end{aligned}$$

where $C_1(m) > 0$ is defined in (2.4).

Recall that by (2.11), a_0 is bounded as long as $J(t) \leq \eta^5$, and that $I(0) \leq K_0\eta^6$ by (2.21) (recall the definition of K_0 in (2.15)). Therefore, under the assumptions that $b_0(t) \geq 0$ and $|t| \leq \eta^{-2}$ we get that as long as $J(t) \leq \eta^5$, we have $J(t) \leq K_1 I(0)$, which gives the result for η sufficiently small, since $I(0) \leq K_0 J(0)$. □

Lemma 2.8. — *Let α_0 be a real with $0 < \alpha_0 \leq \delta$. We suppose that the initial conditions $(a(0), b(0))$ satisfy $0 < a_0(0) < \alpha_0$ and $b_0(0) \geq 0$. Then there exist times $\tau(\alpha_0)$ and τ_η such that*

$$a_0(\tau(\alpha_0)) = \alpha_0, \; a_0(\tau_\eta) = \delta \quad \text{and} \quad b_0(t) > 0 \;\; \text{for} \;\; t \in]0, \tau_\eta].$$

Moreover, we have the estimates

$$\tau(\alpha_0) \leq \tau_\eta \leq C \ln \frac{1}{\eta},$$

and if $b_0(0) = 0$,

$$c \ln \frac{1}{\eta} \leq \tau_\eta.$$

Proof. — • Firstly, we can go back to the particular case $b_0(0) = 0$: for that, we prove that there exists a time $t \in]0, \eta^{-2}[$ such that $b_0(-t) = 0$. We proceed by contradiction : suppose that $b_0(-t) > 0$ for all $t \in [0, \frac{4b_0(0)}{m^2\eta}]$. We know from Lemma 2.7, that for all those t, $J(-t) \leq \eta^5$. Thus $q_0(-t) \leq C\eta^5$, and by (2.11), we have that $a_0(-t) > \eta/2$. Hence,

$$\dot{b}_0(-t) = -f'(a_0(-t)) - q_0(t) > -f'(\frac{\eta}{2}) + O(\eta^5) = m^2\frac{\eta}{2} + O(\eta^3) > m^2\frac{\eta}{3}.$$

Then for all t in $[0, \frac{4b_0(0)}{m^2\eta}]$,

$$b_0(-t) < b_0(0) - tm^2\frac{\eta}{3}$$

holds. In particular, we obtain $b_0(-\frac{4b_0(0)}{m^2\eta}) < 0$: the contradiction.

• So for all initial conditions satisfying the assumption of the lemma, there exists a negative time $-t$ such that $b_0(-t) = 0$ and $J(-t) \leq \eta^5$.

In the following we only consider the case

$$a_0(0) = \eta \quad \text{and} \quad b_0(0) = 0.$$

Observe that the upper bound we will get for τ_η in this case will be an upper bound in the general case.

• Let φ be the function defined by

$$\eta^5\varphi(t) = -2(J(t) - J(0)) - 2(r(t) - r(0)).$$

From the preservation of energy, see (2.2) and (2.9), we get that φ satisfies

$$b_0^2 + f(a_0) = f(\eta) + \eta^5\varphi(t). \tag{2.22}$$

Given that $b_0(0) = 0$ and $\dot{b}_0(0) > 0$ for η sufficiently small, $b_0(t) > 0$ for some times, and as long as $b_0(t) \geq 0$, $b_0(t)$ reads

$$\dot{a}_0(t) = b_0(t) = \sqrt{f(\eta) - f(a_0(t)) + \eta^5\varphi(t)}. \tag{2.23}$$

In particular, as long as $b_0(t) > 0$, by integration and the change of variables $\alpha = a_0(t)$ we obtain

$$t = \int_0^t \frac{\dot{a}_0(s)\mathrm{d}s}{\sqrt{f(\eta) - f(a_0(s)) + \eta^5\varphi(s)}} = \int_\eta^{a_0(t)} \frac{\mathrm{d}\alpha}{\sqrt{f(\eta) - f(\alpha) + \eta^5\varphi(a_0^{-1}(\alpha))}}. \tag{2.24}$$

• Let us prove that, as long as $J(t) \leq \eta^5$

$$|\varphi(t)| \leq C \min(1, t^2). \tag{2.25}$$

Since $J \leq \eta^5$ and by (2.15) together with (2.18), we have $|\varphi(t)| \leq C$. Then, since $b_0(0) = 0$ and $\dot{a}_0(0) = b_0(0) = 0$, from (2.22) we infer that $\dot{\varphi}(0) = 0$. To obtain (2.25), it is therefore enough to prove that $|\ddot{\varphi}| \leq C$. This is done by computing $\ddot{J}$ and $\ddot{r}$ with the relations (2.8), and with Cauchy-Schwarz we can check that $|\ddot{J}| \leq C\eta^5$ and $|\ddot{r}| \leq C\eta^5$.

• We now show that there exist $c, C > 0$ so that as long $J(t) \leq \eta^5$, $a_0(t) \leq \delta$ and $b_0(t) \geq 0$,

$$c(a_0^2 - \eta^2) \leq f(\eta) - f(a_0) + \eta^5\varphi(t) \leq C(a_0^2 - \eta^2). \tag{2.26}$$

Observe that for all $\eta \leq \alpha \leq \delta$,

$$(m^2 - \delta^{2p})(\alpha^2 - \eta^2) \leq f(\eta) - f(\alpha) \leq m^2(\alpha^2 - \eta^2)$$

holds, and recall that $\delta < m^{\frac{1}{p}}$. It is then enough to show that $\eta^5\varphi(t)$ is a negligible amount when η small. Recall the form (2.7) of the system :

$$\ddot{a}_0 = \dot{b}_0 = m^2 a_0 - a_0^{2p+1} - q_0 = -f'(a_0) - q_0(t).$$

On one hand, with the same calculations as in the proof of Lemma 2.5, we obtain that as long as $J(t) \leq \eta^5$, we have $|q_0(t)| \leq \eta^5$. On the other hand, by a study of the variations of f', we get that for $\eta \leq \alpha \leq \delta$, if η small enough then $f'(\alpha) \leq -c\eta$. We deduce that as long as $J(t) \leq \eta^5$ and $a_0 \leq \delta$, we have $\ddot{a}_0 \geq c\eta$ and by Taylor, $|a_0(t) - a_0(0)| \geq c\eta t^2$. This in turn implies that when moreover a_0^{-1} is well-defined (for $b_0(t) > 0$)

$$|a_0^{-1}(\alpha)| = |a_0^{-1}(\alpha) - a_0^{-1}(\eta)| \leq \frac{C}{\eta^{1/2}}|\alpha - \eta|^{1/2}$$

holds. Then by (2.25), we get that

$$\eta^5|\varphi(t)| \leq C\eta^5\eta^{-1}|a_0 - \eta| \leq C\eta^3(a_0^2 - \eta^2),$$

which proves (2.26).

• Firstly, from (2.26) with (2.23), we get that as long as $J(t) \leq \eta^5$ and $a_0(t) \leq \delta^2$, we have $b_0(t) > 0$. Secondly, with (2.26) together with (2.24), we can compute an estimate of t in term of $a_0(t)$. We will see then that $a_0(t := \tau_\eta) = \delta$ holds for a time τ_η smaller than η^{-2}, so that by Lemma 2.7 we are still in the regime $J(t) \leq \eta^5$. Precisely, observe that

$$\int_\eta^{\alpha_0} \frac{\mathrm{d}\alpha}{\sqrt{\alpha^2 - \eta^2}} = \ln\left(\frac{\alpha_0}{\eta} + \sqrt{\frac{\alpha_0^2}{\eta^2} - 1}\right).$$

Then (2.26) together with (2.24) gives the estimate claimed in the lemma, and in particular $\tau_\eta \leq \eta^{-2}$ for η sufficiently small. $\square$

Consider the time τ_η given in Lemma 2.8. We now show that b_0 admits zeros on $]0, +\infty[$ and that the smallest one is close to τ_η.

Lemma 2.9. — *For any initial condition $(a(0), b(0))$ such that $b_0(0) \geq 0$, there exists a time T^1_η such that*

$$b_0(T^1_\eta) = 0, \quad \textit{and} \quad b_0(t) > 0 \textit{ for all } t \in]0, T^1_\eta[.$$

Moreover, if $a_0(0) < \delta$, then there exist C such that

$$0 < T^1_\eta - \tau_\eta \leq C,$$

where τ_η is given by Lemma 2.8 and if $a_0(0) \geq \delta$, then

$$T^1_\eta \leq C.$$

Proof. — Recall that $\delta \in]0, m^{\frac{1}{p}}[$, and δ' is the unique real of $]m^{\frac{1}{p}}, (p+1)^{\frac{1}{2p}} m^{\frac{1}{p}}[$ such that $f(\delta') = f(\delta)$. In the proof we consider the case $a_0(0) < \delta$. To prove the lemma, we proceed in two steps, here is a summary of them :

(i) We show that there exists a time $\tau^1_\eta > \tau_\eta$ such that $b_0(\tau^1_\eta) = b_0(\tau_\eta)$, $a_0(\tau^1_\eta) - \delta' \leq C\eta^5$ and $b_0(t) > b_0(\tau_\eta)$ for $t \in]\tau_\eta, \tau^1_\eta[$. Moreover, $\tau^1_\eta - \tau_\eta \leq C$ holds.

(ii) Then we show the existence of $T^1_\eta > \tau^1_\eta$ such that $b_0(T^1_\eta) = 0$, $|a_0(T^1_\eta) - h(0)| \leq C\eta^2$, and $b_0(t) > 0$ for all $t \in]\tau^1_\eta, T^1_\eta[$. Moreover, $T^1_\eta - \tau^1_\eta \leq C$ holds.

If $a_0(0) \in]\delta, \delta'[$, then in step (i) the proof is the same by considering $a_0(0)$ instead of $a_0(\tau_\eta)$. If $a_0(0) > \delta'$, then step (ii) is sufficient, considering here $a_0(0)$ instead of $a_0(\tau^1_\eta)$.

Step (i). From Lemma 2.8 we know that $a_0(\tau_\eta) = \delta$. We then obtain that $b'_0(\tau_\eta), b_0(\tau_\eta) \geq c_0 > 0$ (c_0 depends only on δ which is fixed). In particular, $b_0(t) \geq b_0(\tau_\eta)$ for small times $t \geq \tau_\eta$. Let us prove this step, by contradiction : suppose that for all t in $]\tau_\eta, \tau_\eta + (h(0) - \delta)/c_0]$, we have $b_0(t) > b_0(\tau_\eta)$. Then for all those t,

$$a_0(t) > \delta + c_0 \cdot (t - \tau_\eta).$$

Then we have $a_0(\tau_\eta + (h(0) - \delta)/c_0) > h(0)$, and thus there exists $t \in]\tau_\eta, \tau_\eta + (h(0) - \delta)/c_0]$ such that $f(a_0(t)) = f(h(0)) = 0$. Hence, for this t, the assumption $b_0(t) > b_0(\tau_\eta)$ is in contradiction with the energy preservation (2.22) for η sufficiently small, given that t is in the regime $|\varphi(t)| \leq C$. Finally, there exists τ^1_η such that $0 < \tau^1_\eta - \tau_\eta \leq C$ and $b_0(\tau^1_\eta) = b_0(\tau_\eta)$.

To get the approximation of $a_0(\tau^1_\eta)$, on the one hand observe that necessarily there exists a time $t \in]\tau_\eta, \tau^1_\eta[$ such that $b'_0(t) = 0$. From (2.7) we thus have $|f'(a_0(t))| \leq C\eta^5$, which means that $|a_0(t) - m^{\frac{1}{p}}| \leq C'\eta^5$. On the other hand, from the preservation of the energy (2.22), given that we are still in the regime $|\varphi(t)| \leq C$, we obtain that $|f(a_0(\tau^1_\eta)) - f(\delta)| \leq C\eta^5$. Then, given that $a_0(\tau^1_\eta) > a_0(t)$, we get that for η sufficiently small, $|a_0(\tau^1_\eta) - \delta'| \leq C\eta^5$.

Step (ii). Let us prove this step by contradiction : suppose that for all t in $]\tau^1_\eta, \tau^1_\eta + 2b_0(\tau_\eta)/f'(\delta)]$, we have $b_0(t) > 0$. Then for all those t, $a_0(t) > a_0(\tau^1_\eta)$ and thus

$$b'_0(t) < -f'(\delta') + O(\eta^5) < \frac{1}{2} f'(\delta) \quad \text{for } \eta \text{ small}.$$

Then we have $b_0(\tau^1_\eta + 2b_0(\tau_\eta)/f'(\delta)) < 0$, and get the contradiction. Thus there exists T^1_η such that $b_0(T^1_\eta) = 0$ and $0 < T^1_\eta - \tau^1_\eta \leq C$. From the energy preservation (2.22) and given that $a_0(T^1_\eta) > a_0(\tau^1_\eta) > m^{\frac{1}{p}}$ and $f'(h(0)) > 0$, we get the estimate on $a_0(T^1_\eta)$. □

Lemma 2.10. — *There exist some constants* $c_0, C_0, C > 0$ *such that for any initial condition* $(a(0), b(0))$ *satisfying*

$$dist\big((a_0(0), b_0(0)), \mathcal{K}_\eta)\big) \le \eta^3, \quad J(0) \le \eta^5,$$

there exists a time T_η *such that*

$$\big|(a_0(T_\eta), b_0(T_\eta)) - (a_0(0), b_0(0))\big| \le C\eta^2, \quad J(T_\eta) \le 2KJ(0) \le \eta^5$$

and

$$c_0 \ln \frac{1}{\eta} \le T_\eta \le C_0 \ln \frac{1}{\eta}.$$

Precisely, we can define uniquely T_η *as the time of first return to the set*

$$\big\{(a, b) \mid a_0 = a_0(0),\ sgn(b_0) = sgn(b_0(0))\big\}, \quad \textit{if}\ \ a_0(0) \in [\delta, \delta']$$

or

$$\big\{(a, b) \mid b_0 = b_0(0),\ sgn(a_0 - m^{\frac{1}{p}}) = sgn(a_0(0) - m^{\frac{1}{p}})\big\}, \quad \textit{if}\ \ a_0(0) \notin [\delta, \delta'].$$

Proof. — Let us explain the strategy of the proof. We follow the periodic orbit with initial $(a(0), b(0))$. When we are $\delta-$close to 0 we use the arguments of Lemma 2.8, and when $a_0 \ge \delta$, we use the arguments of Lemma 2.9.

Notice that by Lemma 2.6 and the precise definition of T_η we know that $(a_0(0), b_0(0))$ and $(a_0(T_\eta), b_0(T_\eta))$ are η^2 close to each other. □

2.5. Many loops : proof of Theorem 1.2 (ii)**.** — We would like to take $(a(T_\eta), b(T_\eta))$ as an initial condition in Lemma 2.10 to iterate the process. But by Lemmas 2.10 and 2.6, we only know that

$$dist\big((a_0(T_\eta), b_0(T_\eta)), \mathcal{K}_\eta\big) \le \eta^2, \quad J(T_\eta) \le K\eta^6.$$

This is a priori too weak to apply Lemma 2.10. But we can choose to apply the Lemma 2.10 with η' such that $(a_0(T_\eta), b_0(T_\eta)) \in \mathcal{K}_{\eta'}$, which gives the same estimates.

We see then that we can iterate the argument as long as $J \le \eta^5$, namely N times (each time corresponds to a loop) with

$$(2K)^N \eta^6 = \eta^5 \quad \text{i.e. } N = c \ln \frac{1}{\eta}.$$

This gives that for $|t| \le c\big(\ln \frac{1}{\eta}\big)^2$, $J \le \eta^5$ and we can iteratively apply Lemma 2.10.

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